[illegible]

Equation of tangent at (x_1, y_1) is

$$\frac{y-y_1}{x-x_1} = \left(\frac{dy}{dx}\right)_{x_1,y_1}$$

Equation of normal at (x_1, y_1) is

$$\frac{y-y_1}{x-x_1} = \frac{-1}{\left(dy/dx\right)_{x_1,y_1}}$$

where $y = mx + c$ is the tangent

then $c^2 = a^2m^2 - b^2$

Problems based on tangent and normal

1. Find the equation of tangent to the hyperbola $4x^2 - 9y^2 = 1$ which is parallel to the line $4y = 5x + 7$

Solution: Since the tangent is parallel to line $4y = 5x + 7$ —— (1)

A line parallel to (1) is

$y = \frac{5}{4}x + c$, $m = \frac{5}{4}$

As equation of hyperbola is $4x^2 - 9y^2 = 1$

$\Rightarrow \frac{x^2}{1/4} - \frac{y^2}{1/9} = 1$

It will be tangent if

$c^2 = \frac{1}{4}\cdot\frac{25}{16} - \frac{1}{9}$ [and $c^2 = a^2m^2 - b^2$]

$c^2 = \frac{25 \times 9 - 16 \times 4}{4 \times 16 \times 9}$

$c^2 = \frac{225 - 64}{4 \times 16 \times 9} = \frac{161}{4 \times 16 \times 9}$

22-07-2020

$\Rightarrow K^2 = \frac{161}{36}$

$\Rightarrow K = \pm \frac{\sqrt{161}}{6}$

Hence the required equation of tangent is

$$4y = 5x + \frac{\sqrt{161}}{6}$$

Problem :→ Find the Locus of the middle points of the portions of the tangents to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ included between the axes.

Solution :→ By the question
Equation of hyperbola is

$$\frac{x^2}{a^2} - y^2/b^2 = 1 \quad \text{——— } (1)$$

The equation of tangent at $\begin{pmatrix} x = a\sec\theta \\ y = b\tan\theta \end{pmatrix}$

is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta - 1 = 0$

$$\Rightarrow \frac{x}{a\cos\theta} + \frac{y}{-b\cot\theta} = 1$$

$\Rightarrow$ Intercept on x- axis $= a\cos\theta$

intercept on y- axis $= -b\cot\theta$

If (α, β) be the middle point of the portion of tangent included between the axes.

22-02-2020

$\alpha = \frac{a\cos\theta + 0}{2}$ $\quad \beta = \frac{0 + (-b\cot\theta)}{2}$

$= \frac{a}{2}\cos\theta$ $\quad = -\frac{b}{2}\cot\theta.$

$\Rightarrow \sec\theta = \frac{a}{2\alpha}$ $\quad \tan\theta = -\frac{b}{2\beta}.$

Since $\sec^2\theta - \tan^2\theta = 1$

$\frac{a^2}{4\alpha^2} - \frac{b^2}{4\beta^2} = 1$

$\Rightarrow \frac{a^2}{\alpha^2} - \frac{b^2}{\beta^2} = 1$

$\therefore$ the Locus of (α, β) is

$\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

Problem Prove that the Locus of poles with respect to the Parabola $y^2 = 4ax$ of the tangents to the hyperbola $x^2 - y^2 = a^2$ is

the ellipse

$4x^2 + y^2 = 4a^2$

Solution :→ Let the pole be (x_1, y_1)

with respect to $y^2 = 4ax$

$\therefore$ Polar of (x_1, y_1) with respect to $y^2 = 4ax$

$yy_1 = 2a(x + x_1)$

$\Rightarrow y = \frac{2a}{y_1}x + \frac{2ax_1}{y_1}$

If it is the tangent to the hyperbola
$x^2-y^2=a^2$ then

$$\frac{4a^2x_1^2}{y_1^2} = a^2\cdot\frac{4a^2}{y_1^2} - a^2 \quad \left[\text{using } c^2 = a^2m^2-a^2\right.$$

$$\Rightarrow 4x_1^2 = 4a^2 - y_1^2$$

$$\Rightarrow 4x_1^2+y_1^2 = 4a^2$$

Thus the locus of the Pole (x_1,y_1) is the required ellipse.

$$4x^2+y^2 = 4a^2$$

<u>Problem</u> :→ If e_1 and e_2 be the ~~eccentrices~~ eccentricities of a hyperbola and its Conjugate
Prove that

$$\frac{1}{e_1^2}+\frac{1}{e_2^2} = 1$$

<u>Solution</u> :- Say & $\frac{x^2}{a^2} - y^2/b^2 = 1$ ① be hyperbola

then $b^2 = a^2(e_1^2-1)$

$$\Rightarrow \frac{b^2}{a^2} = e_1^2-1$$

$$\Rightarrow \frac{a^2+b^2}{a^2} = e_1^2$$

$$\Rightarrow \frac{1}{e_1^2} = \frac{a^2}{a^2+b^2}$$

Now the Conjugate hyperbola of ① is

$$\frac{x^2}{a^2} + \left(-\frac{y^2}{b^2}\right) = -1$$

$$\Rightarrow \frac{x^2}{a^2} - y^2/b^2 = -1$$

$$\text{or } \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

$$\therefore \frac{1}{e_1^2}+\frac{1}{e_2^2} =$$

$$\therefore a^2 = b^2(e_2^2-1)$$

$$\frac{a^2}{b^2}+1 = e_2^2$$

$$\frac{a^2+b^2}{b^2} = e_2^2$$

$$\therefore 1/e_2^2 = \frac{b^2}{a^2+b^2}$$